HAAR-POSITIVE CLOSED SUBSETS OF HAAR-POSITIVE ANALYTIC SETS

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Abstract. We show that every non-Haar-null analytic subset of $\mathbb{Z}^\omega$ contains a non-Haar-null closed subset. Moreover, we also prove that the codes of Haar-null analytic subsets, and, consequently, closed Haar-null sets in the Effros Borel space of $\mathbb{Z}^\omega$ form a $\mathbf{\Delta}^1_2$ set.

It is not hard to see that non-locally-compact Polish groups do not admit a Haar measure (that is, an invariant σ-finite Borel measure). However, Christensen [4] (and later, independently, Hunt-Sauer-Yorke [8]) generalized the ideal of Haar measure zero sets to every Polish group as follows:

Definition 0.1. Let $(G, \cdot)$ be a Polish group and $S \subset G$. We say that S is *Haar-null*, (in symbols, $S \in \mathcal{HN}$) if there exists a universally measurable set $U \supset S$ (that is, a set measurable with respect to every Borel probability measure) and a Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gUh) = 0$. Such a measure μ is called a *witness measure* for S.

This notion has found wide application in diverse areas such as functional analysis, dynamical systems, group theory, geometric measure theory, and analysis (see, e.g., [12, 2, 14, 13, 5, 1]). It provides a well-behaved notion of "almost every" (or "prevalent") element of a Polish group. It is natural to investigate the regularity properties of Haar-null sets. In particular, one might wonder whether "small sets are contained in nice small sets" and whether "large sets contain nice large sets". Concerning the first question, Solecki [13] has shown a positive statement, namely, that every analytic Haar-null set is contained in a Borel Haar-null set. On the negative side, the first and the third author [6] proved that, unlike the situation in locally-compact groups,

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in non-locally compact abelian Polish groups there are Borel Haar-null sets that have no G_δ Haar-null supersets.

In this paper we address the second question, and answer it positively in the case of a concrete non-locally compact Polish group, $\mathbb{Z}^\omega$, that is, the ω'th power of the additive group of the integers:

Theorem 0.2. *Every analytic non-Haar-null subset of $\mathbb{Z}^\omega$ contains a closed non-Haar-null subset.*

Our proof is based on the results of Solecki [13] and Brendle-Hjorth-Spinas [3]. Roughly speaking, a theorem from the former paper allows us to use witness measures of a very special form, and thus to reduce the understanding of the Haar-null ideal to the understanding of the non-dominating ideal (see Section 1 for the definitions), while the latter contains the regularity properties of the latter ideal. The reduction is based on a coding map and utilizes a compactness argument.

We also calculate the exact projective class of the codes of the analytic Haar-null subsets of $\mathbb{Z}^\omega$, as well as the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$, which turn out to be $\mathbf{\Delta}^1_2$.

1. Preliminaries and basic facts

We start with the most important definitions and theorems that will be used in the proof. We will adapt the notation from [9] for descriptive set theoretic concepts.

The following fact is just a trivial consequence of standard results.

Fact 1.1. *Assume that X, Y are Polish spaces, $F \subset X \times Y$ is Borel, μ is a Borel probability measure on X, and $K_0 \subset \mathrm{proj}_X(F)$ is a compact set with $\mu(K_0) > 0$. Then there exists a compact set $K \subset F$ such that $\mathrm{proj}_X(K) \subset K_0$ and $\mu(\mathrm{proj}_X(K)) > 0$.*

Proof. Using the Jankov, von-Neumann Uniformization theorem (see, [9, Theorem 18.1]) there exists a measurable function $h : K_0 \to Y$ with $\mathrm{graph}(h) \subset F$. Consequently, by Lusin's theorem, there exists a compact set $K_1 \subset K_0$ with $\mu(K_1) > 0$ and such that $h \restriction K_1$ is continuous. But then $K = \mathrm{graph}(h \restriction K_1)$ satisfies the required properties. $\square$

If $b \in \omega^\omega$ let us denote by μ_b the natural product probability measure on $\prod_{n\in\omega}[0, b(n)] \subset \mathbb{Z}^\omega$. For a Polish space X we will denote by $\mathcal{K}(X)$ and $\mathcal{F}(X)$ the space of compact subsets of X with the Hausdorff metric and the space of closed subsets of X with the Effros Borel structure, respectively. The following, easy to prove statements will be used:

Fact 1.2. *Let X be a Polish space and $F \subset X$ be closed. Then*

(1) the map $\omega^\omega \to \mathcal{P}(\mathbb{Z}^\omega)$ (that is, the Polish space of the Borel probability measures on $\mathbb{Z}^\omega$) defined by $b \mapsto \mu_b$
(2) the map $\omega^\omega \times \mathcal{K}(\mathbb{Z}^\omega) \to \mathbb{R}$ defined by $(b, K) \mapsto \mu_b(K)$
(3) the map $\mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X)$ defined by $K \mapsto \mathrm{proj}_X(K)$
(4) the set $\{K \in \mathcal{K}(X) : K \subset F\}$

is Borel.

For $f, g \in \omega^\omega$ we will write $f \leq^* g$ if $f(n) \leq g(n)$ holds for each $n \in \omega$ with finitely many exceptions. Recall that a set $S \subset \omega^\omega$ is *dominating* if it is cofinal in $\leq^*$. The σ-ideal of non-dominating sets is denoted by $\mathcal{ND}$.

The following fact, essentially proved in [3], will play a crucial role.

Fact 1.3. *Assume that $A \in \mathbf{\Sigma}^1_1(\omega^\omega)$ is a dominating set and $f : A \to \omega^\omega$ is a Borel function. Then there exists a closed set $C \subset A$ such that $C \notin \mathcal{ND}$ and $f \restriction C$ is continuous.*

Sketch of the proof. Let $W = \{w_\sigma, s_\sigma : \sigma \in \omega^{<\omega}\}$ be a collection with the following properties: $w_\sigma \subset \omega$ and $\mathrm{dom}(s_\emptyset) \subset \omega$ are finite, $s_\emptyset : \mathrm{dom}(s_\emptyset) \to \omega$, and for $\sigma \neq \emptyset$ we have $s_\sigma : w_{\sigma \restriction lh(\sigma)-1} \to \omega$ and for all $i \in w_{\sigma \restriction lh(\sigma)-1}$ $s_\sigma(i) > \sigma(lh(\sigma) - 1)$, finally, for every $x \in \omega^\omega$ we have that $\omega = \mathrm{dom}(s_\emptyset) \cup \bigcup_n w_{x \restriction n}$, where the union is disjoint. If $T \subset \omega^{<\omega}$ is a tree, define $C_{T,W} =$

$$\{y \in \omega^\omega : s_\emptyset \subset y \text{ and } \exists x \in [T]\ \forall n \in \omega (y \restriction w_{x \restriction n} = s_{x \restriction n+1})\}.$$

It is easy to see that the map $\phi_{T,W} : [T] \to C_{T,W}$ defined by assigning to $x \in [T]$ the unique $y \in C_{T,W}$ with the property that $\forall n \in \omega (y \restriction w_{x \restriction n} = s_{x \restriction n+1})$ is continuous and open. Moreover, one can also check that the set $C_{T,W}$ is closed for every T and W.

A *Laver-tree* is a subtree T of $\omega^{<\omega}$ such that it has a stem s (that is, a maximal node with $\forall t \in T$ $(t \subset s \vee s \subset t)$), and for every $t \not\subset s$ from T the set $\{n \in \omega : t ^\frown (n) \in T\}$ is infinite.

In [3] it is shown that A contains a set of the form $C_{\omega^{<\omega},W}$. Consider now the Borel map $f \circ \phi_{\omega^{<\omega},W} : \omega^\omega \to \omega^\omega$. By [7, Example 3.7] there exists a Laver-tree $T \subset \omega^{<\omega}$ such that $f \circ \phi_{\omega^{<\omega},W} \restriction [T]$ is continuous. Clearly, $\phi_{T,W} = \phi_{\omega^{<\omega},W} \restriction [T]$, and since this map is open, $f \restriction \phi_{T,W}([T])(= C_{T,W})$ is continuous. Thus, it is enough to check that the set $C_{T,W}$ is dominating.

Let $z \in \omega^\omega$ be arbitrary. It is not hard to define an $x \in T$ such that $\phi_{T,W}(x) \geq^* z$ inductively: indeed, for every large enough n (namely, if $n > lh(s)$, where s is the stem of T), if $x \restriction n$ is defined, we can find an m so that $m > \max\{z(i) : i \in w_{x \restriction n}\}$ and $x \restriction n ^\frown (m) \in T$. Then for an x obtained this way, it follows from $\forall i \in w_{x \restriction n}$ $(m < s_{x \restriction n+1}(i))$ that whenever n is large enough and $i \in w_{x\restriction n}$ then $\phi_{T,W}(x)(i) > z(i)$. The fact that $|w_\sigma| < \aleph_0$ implies that $\phi_{T,W}(x)$ dominates z. $\square$

We will also make use of another consequence of the results in [3]:

Fact 1.4. *Let $A \subset \omega^\omega \times \omega^\omega$ be a $\mathbf{\Sigma}^1_1$ set. Then the set $S_{\mathcal{ND}} = \{x : A_x \in \mathcal{ND}\}$ is $\mathbf{\Delta}^1_2$.*

Proof. Counting the quantifiers in the definition of non-dominating sets gives that the set $S_{\mathcal{ND}}$ is $\mathbf{\Sigma}^1_2$. Now by [3], $x \in S_{\mathcal{ND}}$ holds if and only if $\forall C \in \mathcal{F}(\omega^\omega)$ $(C \in \mathcal{ND} \vee C \not\subset A_x)$. As it has been noted by Solecki, it follows from the construction in [3] that the set $\{C \in \mathcal{F}(\omega^\omega) : C \in \mathcal{ND}\}$ is $\mathbf{\Delta}^1_2$, thus, a straightforward calculation yields that $S_{\mathcal{ND}}$ is $\mathbf{\Pi}^1_2$ as well. $\square$

A connection between $\mathcal{ND}$ and $\mathcal{HN}$ has already been established by Solecki [13]. We will use the following:

Lemma 1.5. *Let $S \subset \mathbb{Z}^\omega$ be a Haar-null set. There exists a $b \in \omega^\omega$ such that for every $b' \geq^* b$ we have that $\mu_{b'}$ is a witness for $S \in \mathcal{HN}$.*

Let us remark first that this statement has been implicitly proved in [13] and used without proof in [2]. We will indicate how to show it using a slightly different argument from [11].

Sketch of the proof. It is not hard to see that in order to establish the lemma it suffices to produce a b such that for every $b' \geq b$ the measure $\mu_{b'}$ is a witness for $S \in \mathcal{HN}$. Now, one can check that in the proof of [11, Theorem 3.1] only lower bounds are imposed on the sequence $(N(n))_{n\in\omega}$ and consequently on the sequence $(a(n))_{n\in\omega}$ as well. Applying this observation and [11, Theorem 3.1] for a Borel Haar-null $B \supset S$, the choice $b = (a(n))_{n\in\omega}$ yields the lemma. $\square$

In this paper solely the group $\mathbb{Z}^\omega$ will be considered, and so we will use the additive notation for the group operation.

For an integer-valued function $f : X \to \mathbb{Z}$ the notation $|f|$ and cf will be used for the function defined by $x \mapsto |f(x)|$ and $x \mapsto cf(x)$ for $x \in X$ and $c \in \mathbb{Z}$. Also we will write $f \leq g$ if for each $x \in X$ we have $f(x) \leq g(x)$.

2. Complexity estimation

As a warm up, we calculate the complexity of the codes of Haar-null analytic subsets of and the complexity of the closed Haar null subsets of $\mathbb{Z}^\omega$ in the Effros Borel space. It has been shown by Solecki [13], see also [10, 15], that the codes for the closed Haar-null subsets, as well as the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ are neither analytic nor co-analytic.

(In fact, $\mathbb{Z}^\omega$ can be replaced by any non-locally compact Polish group admitting a two-sided invariant metric). So the next result is sharp.

Theorem 2.1. *The codes for Haar-null analytic subsets of* $\mathbb{Z}^\omega$ *form a* $\mathbf{\Delta}^1_2$ *subset, which is neither analytic nor co-analytic. More precisely, if* $U \subset \omega^\omega \times \mathbb{Z}^\omega$ *is a* $\mathbf{\Sigma}^1_1$ *set then the set* $S = \{x \in \omega^\omega : U_x \in \mathcal{HN}\}$ *is* $\mathbf{\Delta}^1_2$ *and there are closed sets* U*, for which this set is neither analytic nor co-analytic. Moreover, the set* $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ *is also* $\mathbf{\Delta}^1_2$*.*

Proof. As mentioned above, by Solecki's results it is enough to prove that S is $\mathbf{\Delta}^1_2$. By Fact 1.4 it suffices to define a $\mathbf{\Sigma}^1_1$ subset A of $\omega^\omega \times \omega^\omega$ with the property that $x \in S \iff A_x \in \mathcal{ND}$. Let

$$(x, h) \in A \iff \exists g \in \mathbb{Z}^\omega (\mu_h(U_x + g) > 0).$$

It follows from [9, Theorem 29.27] and Fact 1.2 that the set A is $\mathbf{\Sigma}^1_1$.

Let $x \in \omega^\omega$ be arbitrary and assume that $U_x \notin \mathcal{HN}$. We show that in this case $A_x = \omega^\omega$. Indeed, for any $h \in \omega^\omega$ the condition $U_x \notin \mathcal{HN}$ implies the existence of a $g \in \mathbb{Z}^\omega$ with $\mu_h(U_x + g) > 0$.

Now assume that $U_x \in \mathcal{HN}$, and towards a contradiction suppose that $A_x \notin \mathcal{ND}$. Then, we apply Lemma 1.5 to U_x and get a $b \in \omega^\omega$. Pick an $h \in A_x$ such that $h \geq^* b$. Then on the one hand μ_h should witness that U_x is Haar-null, on the other hand $\mu_h(U_x + g) > 0$ for some $g \in \mathbb{Z}^\omega$, a contradiction.

To see that the above argument implies that the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ is $\mathbf{\Delta}^1_2$, just fix a Borel isomorphism ι between ω^ω and $\mathcal{F}(\mathbb{Z}^\omega)$. It is straightforward to check that the set $\{(C, h) \in \mathcal{F}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : h \in C\}$ is Borel and so is the set $B = \{(\iota(C), h) \in \mathcal{F}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : h \in C\}$. Now, using the first part of the statement, the set $\{x \in \omega^\omega : \iota^{-1}(x) \in \mathcal{HN}\} = \{x \in \omega^\omega : B_x \in \mathcal{HN}\}$ is $\mathbf{\Delta}^1_2$, and, consequently, its pullback under ι, that is, the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ is $\mathbf{\Delta}^1_2$ as well. $\square$

3. The main result

In this section we prove Theorem 0.2. Let us start with an easy observation.

Lemma 3.1. *Let* $f \in \omega^\omega$ *be arbitrary. Then the set*

$$H(f) = \{g \in \mathbb{Z}^\omega : \exists^\infty n \in \omega \ |g(n)| \leq f(n)\}$$

is Haar-null in $\mathbb{Z}^\omega$*.*

Proof. Let $f'(n) = 2^n(f(n) + 1)$. We will show that the measure $\mu_{f'}$ witnesses that $H(f)$ is Haar-null. Let $h \in \mathbb{Z}^\omega$ be arbitrary. Clearly,

$$H(f) + h = \bigcap_{k \in \omega} \bigcup_{n \geq k} \{g + h : |g(n)| \leq f(n)\},$$

and for every $n \in \omega$ we have

$$\mu_{f'}(\{g+h : |g(n)| \leq f(n)\}) \leq \frac{2f(n)}{f'(n)} \leq \frac{2}{2^n},$$

Thus, using $\sum_{n\in\omega} \frac{2}{2^n} < \infty$ and the Borel-Cantelli lemma, we get that $\mu_{f'}(H(f)+h) = 0$. □

Now we are ready for the proof of the main result. Our strategy will be somewhat similar to the idea of the proof of Theorem 2.1, just significantly more sophisticated. To a given analytic set $A \notin \mathcal{HN}$ we will assign a Borel set D that encodes the witnesses for $A \notin \mathcal{HN}$, i.e., codes for possible witness measures μ and compact sets K, and translations $t \in \mathbb{Z}^\omega$ with $K + t \subset A$ and $\mu(K) > 0$. The coding will be constructed so that it ensures that D is dominating. Using the results of Brendle, Hjorth, and Spinas, we will chose a dominating closed subset of D with some additional properties, and from it a non-Haar-null subset of A will be reconstructed. A compactness argument will yield that this set is in fact closed.

Proof of Theorem 0.2. Let $A \in \mathbf{\Sigma}^1_1(\mathbb{Z}^\omega)$ be a non-Haar null set and let $F \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega)$ with $\mathrm{proj}_{\mathbb{Z}^\omega}(F) = A$. Fix a Borel bijection $\psi : 2^\omega \to \mathcal{K}(\mathbb{Z}^\omega \times \omega^\omega)$ and define a Borel partial mapping $\phi : \omega^\omega \times \mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathcal{K}(\mathbb{Z}^\omega \times \omega^\omega)$ as follows: let $(b,t,c) \in \mathrm{dom}(\phi)$ iff the conjunction of the following holds:

(1) $c - t \in 2^\omega$.
(2) $2b \leq^* |t|$.
(3) $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c-t)) \subset \prod_{n\in\omega}[0, b(n)]$.
(4) $\mu_b(\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c-t))) > 0$.

Let us use the notation $+^p$ for the $(\mathbb{Z}^\omega \times \omega^\omega) \times \mathbb{Z}^\omega \to \mathbb{Z}^\omega \times \omega^\omega$ mapping that is the translation of the first coordinate, i.e., $(r,x) +^p t = (r+t, x)$. Define ϕ for $(b,t,c) \in \mathrm{dom}(\phi)$ by letting

$$\phi(b,t,c) = \psi(c-t) +^p t,$$

in other words, $\phi(b,t,c) = \psi(c-t) +^p t$ is the compact subset of $\mathbb{Z}^\omega \times \omega^\omega$ defined by $\{(r,x) +^p t : (r,x) \in \psi(c-t)\}$.

Finally, we will need a homeomorphism $\mathrm{bij} : \omega^\omega \to \omega^\omega \times \mathbb{Z}^\omega \times \mathbb{Z}^\omega$. In order to be precise, let us fix a concrete one by letting for every $n \in \omega$

- $\mathrm{bij}(f)(0)(n) = f(3n)$,
- for $i \in \{1, 2\}$ define $\mathrm{bij}(f)(i)(n) =$

$$\begin{cases} f(3n+i)/2, & \text{if } f(3n+i) \text{ is even,} \\ -(f(3n+i)+1)/2, & \text{if } f(3n+i) \text{ is odd.} \end{cases}$$

Lemma 3.2. *Let $D = \{f \in \omega^\omega : \phi(\mathrm{bij}(f)) \subset F\}$. Then D is dominating.*

Proof. Assume otherwise, and let $f \in \omega^\omega$ witness this fact. Without loss of generality, we can assume that f is constant on the sets of the form $\{3n, 3n+1, 3n+2\}$, and it attains only positive values. Define an element $f' \in \mathbb{Z}^\omega$ by $f'(n) = 2(f(3n)+1)$.

Using Lemma 3.1 and $A \notin \mathcal{HN}$ we get that $A \setminus H(3f') \notin \mathcal{HN}$. This yields that there exist a compact set $K_0 \subset \prod_n [0, f'(n)]$ with $\mu_{f'}(K_0) > 0$ and a $t \in \mathbb{Z}^\omega$ such that $K_0 + t \subset A \setminus H(3f')$. Now, using Fact 1.1 for the sets $F +^p (-t)$, K_0 and $\mu_{f'}$ we get a compact set $K \subset F +^p (-t)$ (or, equivalently, $K +^p t \subset F$) with $\mu_{f'}(\mathrm{proj}_{\mathbb{Z}^\omega}(K)) > 0$ and $\mathrm{proj}_{\mathbb{Z}^\omega}(K) \subset K_0$.

Consider now the function $g = \mathrm{bij}^{-1}(f', t, t + \psi^{-1}(K))$. We claim that $g \geq^* f$ and $g \in D$, contradicting our initial assumption and thus finishing the proof. In order to see $g \geq^* f$, notice that, as $\emptyset \neq K_0 + t \subset (\prod_n [0, f'(n)] + t) \cap (A \setminus H(3f'))$, necessarily $f' + |t| \geq^* 3f'$, so $|t| \geq^* 2f'$. Then, it is straightforward to check from the definition of bij that $g \geq^* f$ holds.

Checking $g \in D$ is just tracing back the definitions: clearly, $\mathrm{bij}(g) = (f', t, t + \psi^{-1}(K)) \in \mathrm{dom}(\phi)$ holds, as we have already seen (1), (2), and for (3), (4) note that $\psi(t + \psi^{-1}(K) - t) = K$ and $\mathrm{proj}_{\mathbb{Z}^\omega}(K) \subset K_0$. Finally, $\phi((f', t, t + \psi^{-1}(K))) = K +^p t \subset F$. $\square$

Using Fact 1.2 we get that $\mathrm{dom}(\phi)$ is a Borel set, and as $+^p$ is continuous, ϕ is a Borel map. Moreover, using (4) from Fact 1.2, D must be Borel as well. Since non-dominating sets form a σ-ideal, by passing to a dominating Borel subset of D, we can assume that there is an $n_0 \in \omega$ and a sequence $(\beta, \tau, \gamma) \in \omega^{n_0} \times \mathbb{Z}^{n_0} \times \mathbb{Z}^{n_0}$ such that for each $f \in D$ if $\mathrm{bij}(f) = (b, t, c)$ then $2b(k) \leq |t(k)|$ for each $k \geq n_0$ and for each $k < n_0$ we have $\beta(k) = b(k), \tau(k) = t(k), \gamma(k) = c(k)$.

Now Fact 1.3 implies the existence of a closed dominating set $C \subset D$ such that $\phi \circ \mathrm{bij} \restriction C$ is continuous. We claim that the set $C' = \mathrm{proj}_{\mathbb{Z}^\omega}(\bigcup_{x \in C} \phi(\mathrm{bij}(x)))$ is closed and non-Haar null, which finishes the proof, as it is clearly a subset of A.

First, we show that the set is closed. Let $r_n \in C'$ with $r_n \to r$ and assume that $r_n \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b_n, t_n, c_n))$, where $\mathrm{bij}^{-1}(b_n, t_n, c_n) \in C$. Then, $r_n \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b_n, t_n, c_n)) = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c_n - t_n) +^p t_n)$ and by $(b_n, t_n, c_n) \in \mathrm{dom}(\phi)$ we get that $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c_n - t_n)) \subset \prod_k [0, b_n(k)]$ and by our assumptions on D we have $2b_n(k) \leq |t_n(k)|$ for $k \geq n_0$. Then $|r_n(k)| \geq |t_n(k)| - |b_n(k)| \geq |t_n(k)|/2$ and so $2|r_n(k)| + 1 \geq \max\{|b_n(k)|, |t_n(k)|, |c_n(k)|\}$. By our assumptions on the n_0-long initial segments of the elements of $\mathrm{bij}(D)$, and the convergence of r_n we get that the sequence $(b_n, t_n, c_n)_{n\in\omega}$ must contain a convergent subsequence, and, as bij is a homeomorphism, $\mathrm{bij}^{-1}(b_n, t_n, c_n)$ contains such a subsequence as well. If $f \in \omega^\omega$ is its limit then of course $f \in C$, and the continuity of $\mathrm{proj}_{\mathbb{Z}^\omega} \circ \phi \circ \mathrm{bij} \restriction C$ yields that $r \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(\mathrm{bij}(f))) \subset C'$ holds.

Second, assume that C' is Haar-null. By Lemma 1.5 there exists a $b \in \omega^\omega$ such that for each $b' \geq^* b$ the measure $\mu_{b'}$ witnesses $C' \in \mathcal{HN}$. Since the set C is dominating, there exists an $f \in C$ such that $f(3n) \geq b(n)$ holds for every large enough n. Then if $\mathrm{bij}(f) = (b', t', c')$, by definition $b'(n) = f(3n)$, so $\mu_{b'}$ must witness that C' is Haar-null. On the other hand, $\mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b', t', c')) = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t') +^p t') = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t')) + t' \subset C'$, so $\mu_{b'}$ must witness that the set $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t'))$ is Haar-null as well. But, $(b', t', c') \in \mathrm{dom}(\phi)$ holds, so by (4) we have $\mu_{b'}(\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t'))) > 0$, a contradiction. $\square$

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